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MATROID MATCHING IN PSEUDOMODULAR LATTICES

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The matroid matching problem (also known as matroid parity problem) has been intensively studied by several authors. Starting from very special problems, in particular the matching problem and the matroid intersection problem, good characterizations have been obtained for more and more general classes of matroids. The two most recent ones are the class of representable matroids and, later on, the class of algebraic matroids (cf. [4] and [2]). We present a further step of generalization, showing that a good characterization can also be obtained for the class of so-called pseudomodular matroids, introduced by Björner and Lovász (cf. [1]). A small counterexample is included to show that pseudomodularity still does not cover all matroids that behave well with respect to matroid matching.

1. Introduction

The matroid matching problem can be stated as follows. Given a matroid M on some set E and a set $\mathcal{H}$ of lines, i.e. flats of rank 2. Say that a subset $\mathcal{G}$ of $\mathcal{H}$ is *independent* or a *matching,* if

$$r(\mathcal{G}) = 2|\mathcal{G}|.$$

(Here, $r(\mathcal{G})$ denotes the rank of the union of all lines in $\mathcal{G}$.) The matroid matching problem now is to determine (for given M and $\mathcal{H}$) a maximum cardinality matching $\mathcal{G} \subseteqq \mathcal{H}$. The cardinality of a maximum matching is denoted by $\nu(\mathcal{H})$.

In [4] a minimax formula was presented for $\nu(\mathcal{H})$, in case M corresponds to a projective geometry. It was shown there that

$$\nu(\mathcal{H}) = \min r(A) + \sum_{i=1}^{d} \left\lfloor \frac{r(\mathcal{H}_i \cup A) - r(A)}{2} \right\rfloor$$

where the minimum is taken over all partitions

$$\mathcal{H} = \mathcal{H}_1 \dot\cup \ldots \dot\cup \mathcal{H}_d$$

and all flats A of M.

Consider a simple example. Let $\mathcal{H} = \{h_1, h_2, h_3\}$ consist of three concurrent lines and assume $r(\mathcal{H})=3$. Let $p := h_1 \cap h_2 \cap h_3$. Then obviously $\nu(\mathcal{H})=1$ and the minimum is obtained for $A := p$ and $\mathcal{H} = \{h_1\} \dot\cup \{h_2\} \dot\cup \{h_3\}$. This small ex-

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ample already exhibits a fundamental fact: If the above minmax formula is valid in M, then M is such that any three lines which are pairwise coplanar, but all together span a flat of rank 3, must intersect in point. Of course this is true in projective geometrics, due to modularity. It is, however, a fundamental property of a much more general class of matroids, the socalled full algebraic matroids. (These are matroids formed by *all* elements of an algebraically closed field.) In fact, this is the content of the socalled Ingleton—Main Lemma, stated implicitely in [3].

Thus one might expect that the above minmax formula can be extended to full algebraic matroids, and in fact this has been done by A. Dress and L. Lovász [2]. The main result of section 3 shows that it can be extended to a still more general class of matroids, the socalled pseudomodular matroids, introduced by A. Björner and L. Lovász [1].

2. Some Preparations

Lovász's minmax formula for matroid matching is usually proved by induction of the rank of the underlying matroid or the flat spanned by the given set $\mathcal{H}$ of lines. The crucial cases are those where $\mathcal{H}$ forms a socalled double curcuit (cf. [2]):

Definition (cf. [2]).

Let M be a matroid on E and let $A \subseteq E$. Then A is called a double circuit, if $r(A)=r(A \setminus a)=|A|-2$ for every $a \in A$.

For example, consider the "Ingleton—Main Configuration" composed of three lines and let A consist of six points, two on each line. Then A forms a double circuit.

The following is the most important property of double circuits:

Lemma 2.1 (cf. [2]). *Let A be a double circuit. Then there exists a partition*

$$A = A_1 \dot{\cup} \dots \dot{\cup} A_d$$

such that $C_i := A \setminus A_i$ is a circuit for $i=1, \dots, d$ and these are all circuits contained in A.

Sketch of proof. W.l.o.g. let $E=A$. Then M^* is a matroid of rank 2, i.e. a line, and the A_i's correspond to the points on that line. ∎

The partition of A is unique and d is called the *degree* of the double circuit A.

Definition (cf. [2]).

A matroid M on E has the double circuit property, if the following holds: If $A \subseteq E$ is a double circuit of degree d, then there are $d-2$ independent points in E contained in the closure of every circuit in A.

Note, that this again generalizes the Ingleton—Main Property.

Our main result, to be proved in section 3, will be based on the following:

Theorem 2.2 (cf. [2]). *If a matroid has the double circuit property, then Lovász's minmax formula is valid.*

3. The Main Result

Pseudomodular lattices have been introduced in [1]. The reader is referred to this paper for several examples of such lattices, including full algebraic matroids. The original definition of pseudomodular lattices given by Björner and Lovász is somewhat cumbersome and therefore will not be discussed here. However, for *geometric* lattices — and these are of course those we are interested in — pseudomodularity can be defined in a nice way:

Definition (cf. Theorem 1.4 of [1]).

A geometric lattice L is pseudomodular, if the following holds: Let $a, b, c \in L$ and assume that

$$r(a \vee c) - r(a) = r(b \vee c) - r(b) = r(a \vee b \vee c) - r(a \vee b).$$

Then $r((a \vee c) \wedge (b \vee c)) - r(a \wedge b) = r(a \vee c) - r(a)$.

Remark. If we let a, b and c denote the three lines in the Ingleton—Main Configuration, we find that pseudomodularity implies the Ingleton—Main Lemma.

Originally we intended to prove Lovász's minmax formula for pseudomodular geometric lattices in a purely lattice theoretic context. Later on, however, we discovered that a great deal of the proof was already hidden behind Theorem 2.2, due to A. Dress and L. Lovász himself. Therefore, we decided to restrict ourselves to showing that pseudomodularity implies the double circuit property, thereby shortening the proof considerably. On the other hand, it forces us to work both with the lattice theoretic concept of pseudomodularity and the matroid (or set-) theoretic concept of double circuits, which may cause some confusion as far as notational details are concerned. Thus, in what follows, we will use both the lattice theoretical notations such as $\leqq$, $\wedge$, $\vee$ etc. and the set theoretical ones, such as $\subseteqq$, $\cap$, $\cup$ etc. The closure of a set $A \subseteqq E$ will always be denoted by $\bar{A}$ and this will be considered as a flat of the matroid or as an element of the corresponding geometric lattice, depending on the context. We hope that the reader is familiar enough with the relationship between matroids and geometric lattices so that no severe misunderstandings will arise.

In what follows, L will denote a pseudomodular geometric lattice with point set E. M will be the corresponding matroid and $A \subseteqq E$ will be a double circuit of degree d. The partition of A is given by $A = A_1 \dot{\cup} \dots \dot{\cup} A_d$ and $C_i = A \setminus A_i$, $i = 1, \dots, d$ denote the circuits in A. The main theorem we are going to prove is the following

Theorem 3.1.

$$r\Big(\bigcap_{i=1}^{d} \bar{C_i}\Big) \geqq d-2$$

with equality for $d \geqq 3$.

The proof of Theorem 3.1 will be split up into several steps. We start by providing some almost trivial facts, which are not even based on the pseudomodularity of L. Note that Theorem 3.1 is trivially true whenever $d \leqq 2$. Hence we will assume $d \geqq 3$ in what follows.

Lemma 3.2. *Let* $i, j, k \in \{1, \dots, d\}$ *be pairwise distinct. Then*

(i) $$r(\overline{C_i}) = |A| - |A_i| - 1$$

(ii) $$r(\overline{C_i \cap C_j}) = |A| - |A_i| - |A_j|$$

(iii) $$r(\overline{C_i} \vee \overline{C_j}) = |A| - 2$$

(iv) $$\overline{C_i \cap C_j} \vee \overline{C_i \cap C_k} = \overline{C_i}$$

(v) $$\overline{C_i} \wedge \overline{C_j} = \overline{C_i \cap C_j}.$$

Proof. (i) is clear, since C_i is a circuit. (ii) holds, since $C_i \cap C_j$, being a proper subset of C_i, is independent. (iii) follows from the fact that $A \subseteq C_i \cup C_j$ and $r(A) = |A| - 2$. The "$\leqq$" relation in (iv) is trivial. To prove the converse, first note that $C_i = (C_i \cap C_j) \cup (C_i \cap C_k)$, since the A_i's are disjoint. Hence $\overline{C_i} \leqq \overline{C_i \cap C_j} \vee \overline{C_i \cap C_k}$, and the claim follows. Finally, to prove (v), note that

$$r(\overline{C_i} \wedge \overline{C_j}) \leqq r(\overline{C_i}) + r(\overline{C_j}) - r(\overline{C_i} \vee \overline{C_j}) =$$
$$= |A| - |A_i| - |A_j| = r(\overline{C_i \cap C_j}). \quad \blacksquare$$

The following lemma settles Theorem 3.1 in case $d = 3$.

Lemma 3.3. *Let* $i, j, k \in \{1, \dots, d\}$ *be pairwise distinct. Then*

$$r(\overline{C_i} \wedge \overline{C_j} \wedge \overline{C_k}) = |A| - |A_i| - |A_j| - |A_k| + 1.$$

Proof. Let $a := \overline{C_i \cap C_j}$, $b := \overline{C_i \cap C_k}$ and $c := \overline{C_j \cap C_k}$. Then Lemma 3.2 shows that

$$r(a \vee c) - r(a) = r(b \vee c) - r(b) = r(a \vee b \vee c) - r(a \vee b) = |A_i| - 1.$$

Thus, by pseudomodularity of L, we get

$$r((a \vee c) \wedge (b \vee c)) - r(a \wedge b) = r(a \vee c) - r(a) = |A_i| - 1.$$

By Lemma 3.2, this implies

$$r(\overline{C_i} \wedge \overline{C_j} \wedge \overline{C_k}) = r(a \wedge b) = r((a \vee c) \wedge (b \vee c)) - |A_i| + 1 =$$
$$= r(\overline{C_j} \wedge \overline{C_k}) - |A_i| + 1 =$$
$$= |A| - |A_i| - |A_j| - |A_k| + 1. \quad \blacksquare$$

In the next step, we use induction, in order to increase the set $T = \{i, j, k\}$ of indices to more than three.

Lemma 3.4. *Let* $T \subseteq \{1, \dots, d\}$ *contain at least two elements. Then*

(i) $$r\Big(\bigcap_{t \in T} \overline{C_t}\Big) = |A| - \sum_{t \in T} |A_t| + |T| - 2.$$

Furthermore, for any two elements $i, j \in T$ *we have*

(ii) $$\bigcap_{t \in T \setminus i} \overline{C_t} \vee \bigcap_{t \in T \setminus j} \overline{C_t} = \bigcap_{t \in T \setminus \{i, j\}} \overline{C_t}.$$

Proof. For $|T| \leqq 3$, the claim follows immediately from Lemma 3.2 and 3.3. Thus let $|T| > 3$.

Let us introduce the following notation: For $I \subseteq T$ let $R(I) := \bigcap_{t \in T \setminus I} \overline{C_t}$ and let $r(I)$ denote the rank of $R(I)$. If $I = \{i\}$ or $I = \{i, j\}$, we will simply write $R(i)$, $R(i, j)$ etc.

The interesting part of the whole configuration is sketched in fig. 1. below.

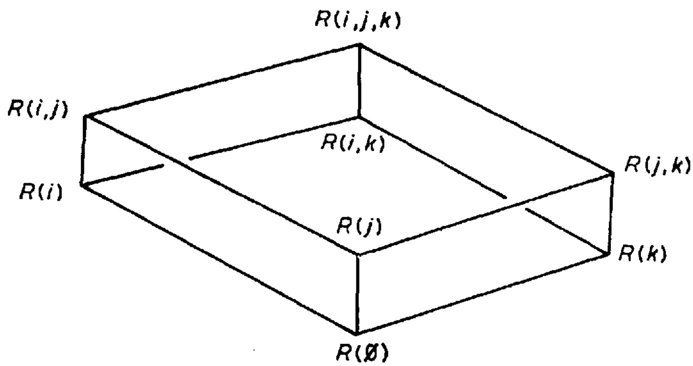

Fig. 1

By induction on $|T|$ we may assume that

$$r(i) = |A| - \sum_{t \in T \setminus i} |A_t| + |T \setminus i| - 2, \text{ etc.}$$

$$r(i, j) = |A| - \sum_{t \in T \setminus \{i, j\}} |A_t| + |T \setminus \{i, j\}| - 2, \text{ etc.}$$

$$r(i, j, k) = |A| - \sum_{t \in T \setminus \{i, j, k\}} |A_t| + |T \setminus \{i, j, k\}| - 2.$$

Furthermore, inductive assumption implies that $R(i, j) \vee R(i, k) = R(i, j, k)$. (Here, the inductive assumption is applied to $T' := T \setminus i$.)

We have to show that

(i) $$r(\emptyset) = |A| - \sum_{t \in T} |A_t| + |T| - 2$$

and

(ii) $$R(i) \vee R(j) = R(i, j), \quad \text{etc.}$$

First note, that the "upper part" of the configuration shown in fig. 1. is modular. More precisely, we have

$$R(i, j) \vee R(i, k) = R(i, j, k),$$

$$R(i, j) \wedge R(i, k) = R(i)$$

and

$$r(i, j) + r(j, k) = r(i, j, k) + r(i), \quad \text{etc.}$$

We will show that the "lower part" is modular, too. We start by showing that $R(i)\vee R(j)=R(i,j)$.

First note, that $R(i,j)$ and $\overline{C_i}$ are modular. In fact, $A\setminus A_i\subseteq\overline{C_i}$ and $A_i\subseteq\subseteq R(i,j)$. Thus $\overline{C_i}\vee R(i,j)=\bar{A}$. This shows that

$$\begin{aligned}r(\overline{C_i}\wedge R(i,j))+r(\overline{C_i}\vee R(i,j)) &= r(j)+r(\bar{A}) =\\ &= |A|-\sum_{t\in T\setminus j}|A_t|+|T\setminus j|-2+|A|-2 =\\ &= |A|-\sum_{t\in T\setminus\{i,j\}}|A_t|+|T\setminus\{i,j\}|-2+|A|-|A_i|-1 =\\ &= r(i,j)+r(\overline{C_i}).\end{aligned}$$

Hence

(1) $$r(\overline{C_i}) = |A|-2+r(j)-r(i,j).$$

On the other hand, $R(i)\vee R(j)\vee\overline{C_i}=\bar{A}$, since $A\setminus A_i\subseteq\overline{C_i}$ and $A_i\subseteq R(i)$. By submodularity, we get

$$\begin{aligned}r(R(i)\vee R(j)) &\geqq r((R(i)\vee R(j))\vee\overline{C_i})+r((R(i)\vee R(j))\wedge\overline{C_i})-r(\overline{C_i}) =\\ &\overset{(1)}{=} |A|-2+r((R(i)\vee R(j))\wedge\overline{C_i})-[|A|-2+r(j)-r(i,j)] \geqq\\ &\geqq |A|-2+r(j)-[|A|-2+r(j)-r(i,j)] =\\ &= r(i,j).\end{aligned}$$

The trivial relation $R(i)\vee R(j)\leqq R(i,j)$, together with the inequality $r(R(i)\vee R(j))\geqq r(i,j)$ implies (ii).

Now let $a:=R(i)$, $b:=R(j)$ and $c:=R(k)$. Then the inductive assumptions shows that

$$r(a\vee c)-r(a) = r(b\vee c)-r(b) = r(a\vee b\vee c)-r(a\vee b) = |A_k|-1.$$

Hence, by pseudomodularity, we conclude that

$$r(R(i,k)\wedge R(j,k))-r(\emptyset) = r(i,k)-r(i).$$

Since $R(i,k)\wedge R(j,k)=R(k)$, we get

$$r(\emptyset) = r(i)+r(k)-r(i,k) = |A|+\sum_{t\in T}|A_t|+|T|-2. \quad \blacksquare$$

If we set $T:=\{1,\ldots,d\}$ in Lemma 3.4, we see that the claim of Theorem 3.1 holds with equality.

4. A Counterexample

In section 3 we have seen that pseudomodularity implies Lovász's minmax relation for matroid matching. Compared with the earlier result of Dress and Lovász on full algebraic matroids, we think that the most interesting point about our result is that it relates a purely lattice theoretical property, namely pseudomodularity, to Lovász's minmax formula (which is a lattice theoretical property, too). Note that, in contrast to pseudomodular lattices (and projective geometries), no lattice characterization for algebraic matroids is known so far.

A natural question that comes up next, is whether Lovász's minmax formula also implies pseudomodularity. In the following we shall present a small (in fact: the smallest possible) counterexample, showing that this is — unfortunately — not true in general.

Example 4.1. We construct an eight point geometry, which is not pseudomodular, but does have the double circuit property.

Let $A:=\{a_1, a_2, a_3\}$, $B:=\{b_1, b_2, b_3\}$ and $C:=\{c_1, c_2\}$. Let $E:=A\cup B\cup C$. All 4-point subsets of E shall be independent and all 6-point subsets shall be dependent. A 5-point subset shall be independent if and only if it meets each of A, B and C.

One readily veryfies that this actually defines a matroid (of rank 5). Let L denote the corresponding geometric lattice.

Now let $a:=A$, $b:=B$ and $c:=C$. Then

$$r(a\vee c)-r(a) = r(b\vee c)-r(b) = r(a\vee b\vee c)-r(a\vee b) = 1.$$

Hence pseudomodularity would imply that

$$r((a\vee c)\wedge(b\vee c))-r(a\wedge b) = r(a\vee c)-r(a).$$

Since $a\vee c=A\cup C$ and $b\vee c=B\cup C$, this reads as

$$2-0=4-3,$$

a contradiction.

Hence L is not pseudomodular.

Next, let us investigate the double circuits of L. There are precisely 7 of them:

$$A\cup B,\quad E\setminus\{a_i\} \quad\text{and}\quad E\setminus\{b_i\} \quad\text{for}\quad i=1,2,3.$$

First, let us deal with $A\cup B$. Every 5-point subset of $A\cup B$ is a circuit. Thus $A\cup B$ contains precisely 6 circuits, i.e. its degree is $d=6$. The closure of every circuit in $A\cup B$ is $A\cup B$, hence the intersection of the closures of all circuits in $A\cup B$ has rank

$$r(A\cup B)=4=d-2.$$

Next, let us deal with one of the remaining double circuits, say $E\setminus\{a_1\}$.

The circuits in $E\setminus\{a_1\}$ are $B\cup C$, $B\cup\{a_2, a_3\}$ and all six point subsets containing at least two points from each of A, B and C (note that any 6 point subset containing just one point out of A, B or C contains a unique circuit). Hence, there are precisely 5 circuits contained in $E\setminus\{a_1\}$, i.e. $E\setminus\{a_1\}$ has degree $d=5$. The circuits are:

$$B\cup C,\quad B\cup\{a_2, a_3\} \quad\text{and}\quad \{a_2, a_3\}\cup B\setminus\{b_i\}\cup C.$$

Their closures are, resp., $B\cup C$, $B\cup A$ and $A\cup B\cup C$.

The rank of the intersection of these closures is $r(B)=3=d-2$.

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See [2] for further references.

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